

RELATIVISTIC TRANSPORT EQUATIONS WITH GENERALIZED MASS SHELL CONSTRAINTS

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We reexamine the derivation of relativistic transport equations for fermions when conserving the most general spinor structure of the interaction and Green function. Such an extension of the formalism is needed when dealing with *e.g.* spin-polarized nuclear matter or non-parity conserving interactions. It is shown that some earlier derivations can lead to an incomplete description of the evolution of the system even in the case of parity-conserving, spin-saturated systems. The concepts of kinetic equation and mass shell condition have to be extended, in particular both of them acquire a non trivial spinor structure which describe a rich polarization dynamics.

1 Introduction

We investigate how the spinor structure of the distribution function affects the transport properties of relativistic fermions. The generalized Wigner distribution function is the Green's function given by

$$F(x, p) = \int d^4r e^{-ip \cdot r} \langle \bar{\Psi}(x + \frac{r}{2}) \Psi(x - \frac{r}{2}) \rangle. \quad (1)$$

It can be developed on the basis of the Dirac gamma matrices as

$$F(x, p) = fI + f_\mu \gamma^\mu + if_{\mu\nu} \sigma^{\mu\nu} + f_{5\mu} \gamma_5 \gamma^\mu + if_5 \gamma_5. \quad (2)$$

Its 16 components generally obey coupled transport equations (see *e.g.* ^{1,2,3}). The usual Boltzmann equation for a scalar distribution appears only if drastic assumptions are made.

The spin degrees of freedom in the context of nuclear matter, for example, might not have been given all the attention they deserve. Relying on the “spin saturation” argument, the derivation of transport theories for numerical simulations of heavy ion collisions generally assumes that the Wigner function is of the simpler form $F(x, p) = (\gamma^\mu \cdot p_\mu + m)\varphi(x, p)$, which permits to recover

the Boltzmann-Uehling-Uhlenbeck (BUU) kinetic equation. In this way, the only manifestation of the fermionic nature of the nucleons comes from the use of a Fermi-Dirac distribution function in the scalar $\varphi(x, p)$.

While the usual BUU approach is already satisfactory *e.g.* for the description of the gross features of heavy ion collisions, and has provided valuable insights in the interpretation of experimental data, we nevertheless believe that the spinor structure should be examined in more detail. Our motivations are fourfold:

(i) On the basis of a simplified relaxation time picture³, a closed system of coupled equations for the scalar distribution f and a spin distribution $f_{5\mu}$ was derived, and it was suggested that the nuclear plasma formed in a collision could relax more efficiently by exploring intermediate polarized states. On the other hand, the improvements brought lately in highly polarized beams and targets could be used to create a polarized excitation, and bring a new wealth of information on the dynamics of dense fermionic systems.

(ii) It is interesting to remember at this stage the analogy with the case of the spin-polarized helium fluids, as they provide an experimental check of the validity of the nonrelativistic limit of a transport theory with internal degrees of freedom⁵. There, effects such as increase of the viscosity, spin-heat coupling, spin waves in the non-degenerate regime can be observed and their magnitude can be compared to predictions. Similar properties can be calculated in the nuclear system³.

(iii) The interest in such studies has been revived in the context of the study of the chiral transition, using the Nambu-Jona-Lasinio as a toy model for the QCD^{1,4}. In the presence of a pseudoscalar condensate, the assumption $F(x, p) = (\gamma^\mu \cdot p_\mu + m)\varphi(x, p)$ cannot be made, instead a coupled transport equation for the system $f, f_{5\mu}$ is obtained.

(iv) Besides the transport equation, the relativistic formalism provides also a constraint or mass shell equation. In the simple cases where the spin saturation hypothesis is made, and also in the chiral model considered by^{1,4}, this equation takes the trivial form $p^2 = m^2$. In the general case however, the constraint equation also retains a spinor structure. One should therefore be careful to use this constraint and not impose from outside the usual $p^2 = m^2$ condition for all components of the Wigner function.

The derivation of the transport equations can proceed with the standard Green function formalism along the Keldysh contour. In a first step, we will assume that the particles are only interacting *via* the self-consistent mean-field $\Sigma_{MF}(X) = \Sigma$. The transport equations for the Wigner function reduce to ^a:

^aAt this level of approximations, we have the properties: $\gamma^0 F^\dagger \gamma_0 = F$ and $\gamma^0 \Sigma^\dagger \gamma_0 = \Sigma$

$$[(i/2)\gamma.\partial + (\gamma.p - m) - \Sigma e^{\frac{-i\Delta}{2}}]F(x, p) = 0 \quad (3)$$

$$F(x, p)[(i/2)\gamma.\partial - (\gamma.p - m) + e^{\frac{i\Delta}{2}}\Sigma] = 0 \quad (4)$$

Here, Δ is the differential operator $\Delta = \overleftarrow{\partial}_x \cdot \overrightarrow{\partial}_p$. This provides a complete system of differential equations describing the evolution of $F(x, p)$. In principle we could use equations (3,4) as they are. We would like however to bring them in a form which we could readily interpret physically as the Vlasov kinetic equation plus a mass shell constraint.

The mass shell constraint is often considered as obvious, to the extent that some authors simply introduce it by hand at the end of the derivation of the kinetic equation. We would like to point out in this contribution that the mass shell equation is in general not a trivial one, but contains important dynamical and spinor information. In a first part, we show on the example of a simple model with scalar and vector mean fields, how different approaches could lead to different kinetic equations if the mass shell constraint is not taken consistently into account. In a second part, we compare the solution of the stationary problem for pseudoscalar or pseudovector mean fields respectively, and illustrate on these examples that the mass shell constraint acquires a non trivial spinor structure.

2 The σ - ω model of nuclear interactions and the mass term issue

We first consider the QHD-I model of the nuclear interaction⁹ where we have scalar and vector mean fields only $\Sigma = (\Sigma_s + \Sigma_{v\mu}\gamma^\mu)$.

2.1 Vlasov equation from the “quadratic” method

A method to extract “kinetic” and “constraint” equations is that used by *e.g.* Elze et al⁷: The equations (3,4) are put in the form

$$(\gamma.K - \mathcal{M})F = 0 \quad ; \quad F(\gamma.K^\dagger - \mathcal{M}^\dagger) = 0 \quad (5)$$

with $\mathcal{M} = M + iN$, $M = m - \Sigma_s$, $N = \frac{1}{2}\partial_\alpha(\Sigma_s)\nabla^\alpha$, $K^\mu = \Pi^\mu + \frac{i}{2}\mathcal{D}^\mu$, $\Pi^\mu = p^\mu + \Sigma_v^\mu$, $\mathcal{D}_\mu = \partial_\mu - \partial_\alpha\Sigma_{v\mu}\nabla^\alpha$, $\nabla^\alpha = \partial/\partial p_\alpha$. By multiplying on the left hand side by $(\gamma.K + \mathcal{M})$ and on the right hand side by $(\gamma.K^\dagger + \mathcal{M}^\dagger)$, and taking the sum and difference of the resulting equations, one obtains in the first order of the gradient expansion^b, after some algebra

^bTerms such as $\mathcal{D}.\mathcal{D}/4$ can be neglected. The same argument can be given performing a semiclassical expansion in powers of \hbar

$$[\Pi^\mu \mathcal{D}_\mu - M \partial_\alpha(\Sigma_s) \nabla^\alpha] F + \frac{1}{4} \mathcal{F}_{\mu\nu} [\sigma^{\mu\nu}, F] + \frac{1}{2} \partial_\mu(\Sigma_s) \{\gamma^\mu, F\} = \mathcal{O}^2 \quad (6)$$

$$[\Pi^2 - M^2] F + \frac{i}{4} \mathcal{F}_{\mu\nu} \{\sigma^{\mu\nu}, F\} + \frac{i}{2} \partial_\mu(\Sigma_s) [\gamma^\mu, F] = \mathcal{O}^2 \quad (7)$$

where $\mathcal{F}^{\mu\lambda} = \partial^\mu(\Sigma_v^\lambda) - \partial^\lambda(\Sigma_v^\mu)$. In this way, familiar looking mass shell condition and kinetic equation are recovered. This “quadratic” method however has been criticized¹⁰ on the ground that information could be lost in the process, in the same way as information is lost on the fermionic nature of the particle by going from the Dirac to the Klein Gordon equation. The issue is important since these authors argue that an other – “non-quadratic” – method of derivation yields a different kinetic equation where the mass term $M \partial_\alpha(\Sigma_s) \nabla^\alpha$ is absent. We will return to this issue in the next section. We first analyze more carefully the equations and show that the mass term is also obtained without squaring the Dirac equation.

2.2 A “pedestrian” method

The components of system of equations (3,4) on the basis of Dirac gamma matrices yield in the first order gradient approximation

$$\mathcal{D}_\mu f^\mu + \partial_\lambda(M) \nabla^\lambda f = 0 \quad (8)$$

$$\mathcal{D}_\mu f + 4\Pi^\alpha f_{\alpha\mu} + \partial_\lambda(M) \nabla^\lambda f_\mu = 0 \quad (9)$$

$$-\frac{1}{2} \epsilon^{\mu\nu\alpha\beta} \mathcal{D}_\alpha f_{5\beta} + \Pi^{[\mu} f^{\nu]} - \partial_\lambda(M) \nabla^\lambda f^{\mu\nu} = 0 \quad (10)$$

$$\epsilon^{\mu\nu\alpha\beta} \mathcal{D}_\nu f_{\alpha\beta} - 2\Pi^\mu f_5 + \partial_\lambda(M) \nabla^\lambda f_5^\mu = 0 \quad (11)$$

$$2\Pi_\mu f_5^\mu + \partial_\lambda(M) \nabla^\lambda f_5 = 0 \quad (12)$$

$$\Pi_\mu f^\mu - M f = 0 \quad (13)$$

$$\Pi^\mu f - M f^\mu + \mathcal{D}_\alpha f^{\mu\alpha} = 0 \quad (14)$$

$$\mathcal{D}_{[\mu} f_{\nu]} - 4M f_{\mu\nu} + 2\epsilon_{\mu\nu\alpha\beta} \Pi^\alpha f^{5\beta} = 0 \quad (15)$$

$$\mathcal{D}_\mu f_5 - 2M f_{5\mu} + 2\epsilon_{\mu\nu\alpha\beta} \Pi^\nu f^{\alpha\beta} = 0 \quad (16)$$

$$\mathcal{D}_\mu f_5^\mu + 2M f_5 = 0 \quad (17)$$

where $a^{[\mu} b^{\nu]} = a^\mu b^\nu - a^\nu b^\mu$, $\epsilon^{\mu\nu\alpha\beta}$ is the completely antisymmetrized Levi-Civita tensor and we have defined as before $\Pi^\mu = p^\mu + \Sigma_v^\mu$, $M = m - \Sigma_s$, $\nabla^\lambda = d/dp_\lambda$, $\mathcal{D}_\mu = \partial_\mu - \partial_\lambda \Pi_\mu \nabla^\lambda$.

In order to calculate the physical observables (baryon number current J^μ and energy momentum $T^{\mu\nu}$), only the components $f = Tr(F)$ and $f^\mu = Tr(\gamma^\mu F)$ are necessary. The f^μ component allows to calculate the nucleon and vector field contributions to J^μ and $T^{\mu\nu}$ while f gives the gap equation

and the scalar field contributions⁸. We can obtain a kinetic-like equation by multiplying (9) with Π^μ :

$$\Pi_\mu \partial^\mu f - \partial_\mu(M) f^\mu - [\Pi_\mu \partial_\lambda(\Pi^\mu) - M \partial_\lambda(M)] \nabla^\lambda f = 0. \quad (18)$$

A kinetic equation for f^μ can be derived from (9) and (15) as follows. We multiply (9) with M and (15) by Π^μ and sum the resulting two equations in order to eliminate the $f^{\mu\nu}$ contribution. We obtain in this way

$$\Pi_\alpha \partial^\alpha f^\beta - \partial^\beta(M) f - [\Pi_\alpha \partial_\lambda(\Pi^\alpha) - M \partial_\lambda(M)] \nabla^\lambda f^\beta + [\partial^\beta \Pi^\alpha - \partial^\alpha \Pi^\beta] f_\alpha = 0. \quad (19)$$

Equations (18) and (19) can be rewritten as

$$[\Pi^\alpha \mathcal{D}_\alpha - M \partial_\alpha(\Sigma_s) \nabla^\alpha] f + \partial_\mu(\Sigma_s) f^\mu = 0 \quad (20)$$

$$[\Pi^\alpha \mathcal{D}_\alpha - M \partial_\alpha(\Sigma_s) \nabla^\alpha] f^\mu + \mathcal{F}^{\mu\lambda} f_\lambda = 0 \quad (21)$$

It is straightforward to see that these equations are exactly what one obtains from taking the trace of (6) with I and γ^μ respectively. As for the mass shell equations, we obtain the one for f by multiplying (14) by Π^μ and using (13)

$$(\Pi^2 - M^2) f + \Pi_\mu \mathcal{D}_\nu f^{\mu\nu} = 0 \quad (22)$$

and the one for f^μ by multiplying (10) with Π^μ making use of (13) and (14)

$$(\Pi^2 - M^2) f^\mu + (1/2) \epsilon^{\mu\alpha\beta\lambda} \Pi_\alpha \mathcal{D}_\beta f_{5\lambda} + M \mathcal{D}_\alpha f^{\mu\alpha} - \partial_\lambda(\Sigma_s) \Pi_\nu \nabla^\lambda f^{\mu\nu} = 0. \quad (23)$$

We notice that (22,23) differs from the traces of (7) with I and γ^μ .

We now proceed to show how can one recover the equations of Elze. Let us apply derivatives^c to (9), obtaining $\Pi_\mu \mathcal{D}_\nu f^{\mu\nu} = \partial_\mu(\Sigma_{\nu\nu}) f^{\mu\nu} + \mathcal{O}^2$ and $\partial_\lambda(\Sigma_s) \Pi_\nu \nabla^\lambda f^{\mu\nu} = -\partial_\lambda(\Sigma_s) f^{\mu\lambda} + \mathcal{O}^2$; and to (15), obtaining $\epsilon^{\mu\alpha\beta\lambda} \Pi_\alpha \mathcal{D}_\beta f_{5\lambda} + 2M \mathcal{D}_\alpha f^{\mu\alpha} = \epsilon^{\mu\alpha\beta\lambda} \mathcal{D}_\alpha(\Pi_\beta) f_{5\lambda} + \mathcal{O}^2$. Substituting in (22,23) we see that the traces with I and γ^μ of the mass shell equation (7) of Elze *et al.* are indeed recovered. The same procedure could be applied to obtain kinetic and mass shell equations for all components of the Wigner function.

Note that (20-23), can also be obtained directly by multiplying (3,4) right and left by $(\gamma \cdot \Pi + M)^d$ and take the sum and difference

$$2(\Pi^2 - M^2) F + \frac{i}{2} \Pi_\mu \mathcal{D}_\nu \{\sigma^{\mu\nu}, F\} + \frac{i}{2} M \mathcal{D}_\mu [\gamma^\mu, F] - \frac{i}{2} \Pi_\mu \partial_\alpha(\Sigma_s) \nabla^\alpha [\gamma^\mu, F] = 0 \quad (24)$$

$$[\Pi^\mu \mathcal{D}_\mu - M \partial_\alpha(\Sigma_s)] F + \frac{1}{2} \Pi_\mu \mathcal{D}_\nu [\sigma^{\mu\nu}, F] + \frac{1}{2} M \mathcal{D}_\mu \{\gamma^\mu, F\} - \frac{1}{2} \Pi_\mu \partial_\alpha(\Sigma_s) \nabla^\alpha \{\gamma^\mu, F\} = 0. \quad (25)$$

^cOnly at this stage an operation is made that cannot be inverted

^dwithout derivative, one can go back any time to (3,4) multiplying again with $(\gamma \cdot \Pi - M)$

Then, (22) and (23) are the traces of (24) with I and γ^μ .

Until (24,25), we can be fully sure that no information was lost. It is tempting to use (6,7) since these equations take a more compact form and have the nice feature of containing explicitly the precession terms $\mathcal{F}_{\mu\nu}[\sigma^{\mu\nu}, F]$. They will probably give no trouble for all practical purposes. One should however keep in mind that the system (24,25) is more general than (6,7) as some loss of generality is taking place in the derivation process. In any case, we find that the mass term $M\partial_\alpha(\Sigma_s)\nabla^\alpha$ is present in (20,21,25) contrary to the claims of the authors of ¹⁰. We will examine their equations in section (2.3) and show that the mass term discrepancy stems from a faulty mass shell condition.

We now discuss a much more severe restriction commonly imposed on the mass shell, which consists of approximating (7) by $(\Pi^2 - M^2)F = 0$. If f and f^μ are remaining on the mass shell $\Pi^2 - M^2 = 0$, equations (20,21) would represent a self contained system for the evolution of f and f^μ . From (22,23) we see that in general however they deviate from this mass shell by terms involving the derivatives of $f^{\mu\nu}$ and f_5^μ . Only in symmetric nuclear matter in a state of (homogeneous) equilibrium, and when the system is unpolarized, does the stationary solution reduce $F = (\gamma \cdot \Pi + M)\delta(\Pi^2 - M^2)\varphi$. Unfortunately, very often it is assumed that the spinor structure remains unchanged out of equilibrium, ($F = f + f_\mu\gamma^\mu$, $0 = f^{\mu\nu} = f_5 = f_{5\mu}$). Then the system (8-17) simplifies, namely: Eqs. (11,12,16,17) contain no information. Due to the assumption $f^{\mu\nu} = 0$, Eq. (14) gives $f^\mu = (\Pi^\mu/M)f$ while Eq. (13) returns the mass shell condition $(\Pi^2 - M^2)f = 0$. Eq. (10) gives $\Pi^{[\mu}f^{\nu]} = 0$ which is trivial with $f^\mu = (\Pi^\mu/M)f$. We are left with Eqs. (8,9,14), and also (18,19) which were derived from the former. Then, either

- * by replacing f^μ by $(\Pi^\mu/M)f$ in Eq. (8),
 - * or multiplying (9) by Π^μ and replacing f^μ ,
 - * or replacing f^μ in Eq. (19) which was obtained from (9) and (15),
- we arrive at

$$\left\{ \Pi^\mu \partial_\mu - [\Pi^\mu \partial_\lambda (\Pi_\mu) - M \partial_\lambda (M) \nabla^\lambda] \right\} \left(\frac{f}{M} \right) = 0. \quad (26)$$

This equation might be further transformed by taking the change of variables $(x, p) \rightarrow (x, \Pi)$ so that we obtain

$$\left\{ \Pi^\mu \partial_\mu - \Pi^\mu \mathcal{F}^{\mu\lambda} \frac{\partial}{\partial \Pi^\lambda} - M \partial_\lambda (M) \frac{\partial}{\partial \Pi^\lambda} \right\} \left(\frac{f}{M} \right) = 0. \quad (27)$$

with $\mathcal{F}^{\mu\lambda} = \partial^\mu(\Sigma_v^\lambda) - \partial^\lambda(\Sigma_v^\mu)$. This is the equation which is extensively used in numerical simulations of heavy ion collisions with a Boltzmann-Uehling-Uhlenbeck collision term on the right hand side.

Since Eq. (27) was arrived at by three different methods, we could think that the result is indeed safe. However we still not have used *all* the information available in the system (8-17). For example, we could have replaced f^μ by Π^μ/Mf and take $f^{\mu\nu} = 0$ in Eq. (15) directly and obtain

$$\Pi^{[\mu}\partial^{\nu]}(\frac{f}{M}) - \Pi^{[\mu}\partial_\lambda(\Pi^{\nu]})\nabla^\lambda(\frac{f}{M}) = 0. \quad (28)$$

Strictly speaking, we simply do not have a justification to put $f^{\mu\nu} = 0$. This term has no reason to vanish out of equilibrium. It could for example acquire contributions like $\mathcal{F}^{\mu\nu}f$, $\Pi^{[\mu}\partial^{\nu]}(M)f$, or time-reversal invariance breaking terms like $U^{[\mu}f^{\nu]}$.

2.3 Vlasov equation from the “Zubarev” method

In this alternative derivation¹⁰ (“Zubarev formalism”), after defining a proper time $\tau = x^\mu u_\mu$, one puts the Dirac equation into Hamiltonian form

$$iu^\mu\partial_\mu\psi = i\frac{\partial\psi}{\partial\tau} = H\psi \quad \text{with } H = -i\sigma^{\mu\nu}u_\mu\partial_\nu + m\gamma.u + \gamma.u\Sigma e^{\frac{-i\Delta}{2}}$$

The conjugate equation for $\bar{\psi}$ reads

$$iu^\mu\partial_\mu\bar{\psi} = i\frac{\partial\bar{\psi}}{\partial\tau} = \bar{\psi}\tilde{H} \quad \text{with } \tilde{H} = i\sigma^{\mu\nu}u_\mu\tilde{\partial}_\nu - m\gamma.u - e^{\frac{i\Delta}{2}}\tilde{\Sigma}\gamma.u. \quad (29)$$

We have $\tilde{H} = -\gamma_0 H^\dagger \gamma_0$ and $\tilde{\Sigma} = \gamma_0 \Sigma^\dagger \gamma_0 = \Sigma$. Acting on the definition of F with $u.\partial F = \dots$ and making these replacements, we arrive at^e

$$u.\partial F = \int d^4R e^{-ip.R} \left[-i\bar{\psi}\tilde{H} \otimes \psi - i\bar{\psi} \otimes H\psi \right] = -i(HF + F\tilde{H}). \quad (30)$$

Choosing now $u^\mu = \Pi^\mu/\sqrt{\Pi^2}$, one arrives at the “kinetic equation” of Smolyan-sky *et al.*¹⁰.

$$\begin{aligned} \Pi.\partial F + \frac{1}{2}\Pi_\mu\partial_\nu[\sigma^{\mu\nu}, F] + iM[\gamma.\Pi, F] - \Pi^\mu\partial_\alpha(\Sigma_{v\mu})\nabla^\alpha F \\ - \frac{1}{2}\Pi_\mu\partial_\alpha(\Sigma_{v\nu})[\sigma^{\mu\nu}, \nabla^\alpha F] - \frac{1}{2}\partial_\alpha(\Sigma_s)\{\gamma.\Pi, \nabla^\alpha F\} = 0 \end{aligned} \quad (31)$$

One can easily see that the same equation can be obtained by multiplying the eqs. (3,4) by $\gamma.\Pi$ from the left and from the right and taking the sum.

However, we still have an other equation, obtained by multiplying on the right and left hand side and subtracting, which is the corresponding constraint equation. It reads

^eNote that this can be put in the form of the Von Neumann equation $i\partial_0 F = [H, F]$ if we choose the 0 coordinate along u^μ , so that $\tilde{\Sigma}\gamma.u = \gamma.u\Sigma$ and $\tilde{H} = -H$ in the mean field approximation.

$$\begin{aligned} & \frac{i}{2}\Pi_\mu\partial_\nu\{\sigma^{\mu\nu}, F\} + 2\Pi^2 F - M\{\gamma.\Pi, F\} \\ & - \frac{i}{2}\Pi_\mu\partial_\alpha(\Sigma_{\nu\nu})\{\sigma^{\mu\nu}, \nabla^\alpha F\} - \frac{i}{2}[\gamma.\Pi, \partial_\alpha(\Sigma_s)\nabla^\alpha(F)] = 0 \end{aligned} \quad (32)$$

Until now, the system (31,32) is fully equivalent to the original set of equations (3,4)^f. However, the second “constraint” equation is not taken into account by Smolyansky *et al.* who replace it by the mass shell condition $\Pi^2 = M^2$, although we can see from the explicit expression that the M^2 term is missing and that it contains instead dynamics at the same order as the other equation. Moreover, one can see that in the free case, the well known kinetic equation $p.\partial F = 0$, is not recovered from (32) alone^g. This example illustrates that some care has to be paid that the seemingly obvious mass shell condition should be obtained from the calculation, less some part of the dynamics may be neglected by imposing this condition beforehand.

2.4 Vlasov equation: discussion

There are several ways one can derive an equation of the form $\Pi.\partial F + D_K[F] = 0$, each method of derivation leading to a different set of additional terms $D_K[F]$. To this equation should be associated the corresponding mass shell condition which also contains dynamical effects $(\Pi^2 - M^2)F + D_M[F] = 0$. If the usual mass shell condition were imposed in the form $(\Pi^2 - M^2)F = 0$ from outside, terms $D_M[F]$ of the same order as $D_K[F]$ would be neglected, leading to a wrong description of the dynamics. The Zubarev method is particularly treacherous in this respect, since the neglected terms are of order zero in the gradient expansion. The method developed by Elze *et al.* on the other hand permits to minimize the importance of the $D_M[F]$ terms so that these are of second order in the gradient expansion if the interactions are switched off. When the mean field is present, however, $D_M[F]$ remains of first order and has to be considered.

3 Pseudoscalar vs. Pseudovector mean fields

In order to illustrate the highly nontrivial structure of the mass shell constraint in the general case, we will compare here the stationary, homogeneous solution in the presence of a nonvanishing pseudoscalar or pseudovec-

^fSince we can always recover the initial set of equations by multiplying a second time with $\gamma.\Pi$ and use $(\gamma.\Pi)^2 = \Pi^2$

^gIt can be shown that the equations $p.\partial F = \mathcal{O}^2$ and $(p^2 - m^2)F = \mathcal{O}^2$ can be recovered from the system (31),(32) by submitting it to a series of manipulations which amounts to return back to the system (3,4) by multiplying (31) and (32) by $\gamma.\Pi$, and taking the sum and difference, and then apply the technique exposed in the previous section

tor mean field. The equations (3,4) now read $(\gamma.p - m - \Sigma)F(x, p) = 0 = F(x, p)(\gamma.p - m - \Sigma)$. The solution of these equations is found to be⁶

$$F(x, p) = P_L(x, p) Q(x, p) P_R(x, p) \delta(D) \quad (33)$$

Here, $P_L(x, p)$ and $P_R(x, p)$ are matrices in the Clifford algebra such that $(\gamma.p - m - \Sigma) P_R(x, p) = D = P_L(x, p) (\gamma.p - m - \Sigma)$, $D = \det(\gamma.p - m - \Sigma)$ being a c-number. The Dirac δ function provides for the cancelation of the products $(\gamma.p - m - \Sigma)F$ and $F(\gamma.p - m - \Sigma)$. $Q(x, p)$ is an arbitrary matrix in the Clifford algebra and contains the thermodynamic averages. It can be shown that $P_R = P_L = P$, and P can be interpreted as a projector on the positive-energy state on the mass shell $D = 0$.

3.1 Scalar, vector and pseudoscalar mean fields

We assume here that the mean field self energy is of the form: $\Sigma = (\Sigma_s + \Sigma_{v\mu}\gamma^\mu + \Sigma_5\gamma_5)$. The stationary homogeneous solution to the Vlasov system of equations is now found to be^h

$$\begin{aligned} (p_*^2 - m_*^2 - \Sigma_5^2)f &= 0 \quad ; \quad f^\mu = \frac{p_*^\mu}{m_*}f \quad ; \quad f_5 = \frac{\Sigma_5}{m_*}f \quad ; \\ f_5^\mu &= \frac{1}{m_*}\epsilon^{\mu\nu\rho\lambda}p_\nu^*f_{\rho\lambda} \quad ; \quad f^{\mu\nu} = \frac{\Sigma_5 p_*^{[\mu}f_5^{\nu]} + m_*\epsilon^{\mu\nu\rho\lambda}p_\rho^*f_{5\lambda}}{\Sigma_5^2 + m_*^2} \end{aligned} \quad (34)$$

with again $p_*^\mu = p^\mu + \Sigma^{v\mu}$, $m_* = m - \Sigma_s$. The solution factorizes as

$$F(x, p) = P_\oplus [\Lambda_\oplus \delta(p_*^2 - m_*^2 - \Sigma_5^2) \varphi(x, p)] P_\oplus \quad (35)$$

with generalized definitions of two projection operators: The projector on the positive energy state of the mass shell $p_*^2 - m_*^2 - \Sigma_5^2 = 0$

$$P_\oplus = 1 + \frac{p_*^\mu}{m_*}\gamma_\mu + i\frac{\Sigma_5}{m_*}\gamma_5 \quad (36)$$

and the projector on the local spin direction

$$\Lambda_\oplus = (1 + \gamma_5\gamma_\mu\mathcal{S}^\mu)/2; \quad \mathcal{S}^\mu = \frac{m_*f_5^\mu}{(m_*^2 + \Sigma_5^2)\varphi} \quad (37)$$

with $f = \delta(p_*^2 - m_*^2 - \Sigma_5^2)\varphi$ and the constraint $p_\mu^*\mathcal{S}^\mu = 0$

In this case, we have four independent distribution functions: f and the three components of f_5^μ orthogonal to p_*^μ .

^hThe reader may easily convince himself that inserting the expression for $f^{\mu\nu}$ in that of f_5^μ leads to an identity

3.2 Scalar, vector and pseudovector mean fields

Let's assume that $\Sigma = (\Sigma_s + \Sigma_{v\mu}\gamma^\mu + \Sigma_{5\mu}\gamma_5\gamma^\mu)$. We find that the stationary homogeneous solution of the Vlasov equation contains only one independent distribution function (the scalar one f). The other components are given by

$$f^\mu = \mathcal{A} \frac{\Sigma_{5\mu}}{m_*} + \mathcal{B} \frac{p_*^\mu}{m_*} ; f_5^\mu = \mathcal{C} \frac{\Sigma_{5\mu}}{m_*} + \mathcal{D} \frac{p_*^\mu}{m_*} ; f_5 = 0 ; f^{\mu\nu} = \epsilon^{\mu\nu\rho\lambda} \frac{p_{\rho}^* \Sigma_{5\lambda}}{m_*^2} \mathcal{E} \quad (38)$$

The expressions of $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$ are given in the appendix. The mass shell condition is given by $-m_* f + p_\alpha^* f^\alpha - \Sigma_{5\alpha} f_5^\alpha = 0$ or equivalently $Df = 0$ with

$$\begin{aligned} D &= [(1 - p_*^2/m_*^2 - \Sigma_{5\alpha}\Sigma_5^\alpha/m_*^2)^2 - 4(p_\alpha^*\Sigma_5^\alpha/m_*^2)^2 + 4\Sigma_{5\alpha}\Sigma_5^\alpha/m_*^2] \\ &= m_*^4 - 2m_*^2(p_\alpha^* + \Sigma_{5\alpha})(p_\alpha^* - \Sigma_{5\alpha}) + (p_\alpha^* - \Sigma_{5\alpha})(p_\beta^* - \Sigma_{5\beta})(p_\alpha^* + \Sigma_{5\alpha})(p_\beta^* + \Sigma_{5\beta}) \end{aligned} \quad (39)$$

The solution can again be expressed in a compact form with the help of a generalized projector P_\oplus

$$F(x, p) = P_\oplus \delta(D) \varphi(x, p) \quad (40)$$

$$P_\oplus = \frac{(m_* - p_\mu^* \gamma^\mu - \Sigma_{5\mu} \gamma_5 \gamma^\mu)(m_* - p_\nu^* \gamma^\nu - \Sigma_{5\nu} \gamma_5 \gamma^\nu)(m_* - p_\rho^* \gamma^\rho - \Sigma_{5\rho} \gamma_5 \gamma^\rho)}{2m_*(m_*^2 - p_*^2 + \Sigma_{5\alpha} \Sigma_5^\alpha)}$$

on the mass shell $D = 0$ defined as above (eq. 39). It is obvious that the physics described by the solution in a pseudoscalar field will be very different from the physics in a pseudovector field. In the case of the Walecka model for example, it will thus not be a simple matter of convenience to choose PS or PV coupling of the pion field.

4 Conclusion

On the basis of two examples, we have wished to stress the importance of taking into account the mass shell condition properly. Only in very simple cases does this condition take the simple form $(\Pi^2 - M^2)F = 0$, in general however it is highly non trivial.

In the case of the $\sigma - \omega$ model of nuclear matter, it was shown that one cannot separate clearly the “kinetic” and “mass shell” conditions in general. The mass shell condition always contains some dynamical contribution in the interacting case. Overlooking this fact may even lead to wrong dynamics.

In the case of a system with pseudoscalar or pseudovector coupling, we have solved the stationary case and shown that the mass shell condition acquires a rich spinor structure. The PS or PV coupling lead to very different solutions.

Appendix

The full expression of the coefficients entering the PV result is:

$$\begin{aligned}
\mathcal{A} &= [(p_\alpha^* \Sigma_5^\alpha / m_*^2 (f + p_\alpha^* f^\alpha / m_*) - (p_*^2 / m_*^2 - 1) \Sigma_{5\alpha} p_*^\alpha / m_*^2 f)] / \mathcal{F} \\
\mathcal{B} &= [(p_*^2 / m_*^2 - 1) (\Sigma_{5\alpha} f_5^\alpha / m_* - f) - (\Sigma_{5\alpha} p_*^\alpha / m_*^2)^2 f] / \mathcal{F} \\
\mathcal{C} &= [(1 + \Sigma_{5\alpha} \Sigma_5^\alpha / m_*^2) (f + p_\alpha^* f^\alpha / m_*) - (\Sigma_{5\alpha} p_*^\alpha / m_*^2)^2 f] / \mathcal{F} \\
\mathcal{D} &= [(\Sigma_{5\alpha} p_*^\alpha / m_*^2) \Sigma_{5\alpha} f_5^\alpha / m_* - 2f - \Sigma_{5\alpha} \Sigma_5^\alpha / m_*^2 f] / \mathcal{F} \\
\mathcal{E} &= -(\mathcal{A} + \mathcal{D}) / 2 \\
&= [(1 + \Sigma_{5\alpha} \Sigma_5^\alpha / m_*^2) (1 + (\Sigma_{5\alpha} p_*^\alpha / m_*^2)^2 - p_*^2 / m_*^2)] / (\mathcal{F} \mathcal{G}) \\
\mathcal{F} &= (\Sigma_{5\alpha} p_*^\alpha / m_*^2)^2 + (1 - p_*^2 / m_*^2) (1 + \Sigma_{5\alpha} \Sigma_5^\alpha / m_*^2) \\
\Sigma_{5\alpha} f^\alpha &= [2(p_\alpha^* \Sigma_5^\alpha / m_*^2)^2 - \Sigma_{5\alpha} \Sigma_5^\alpha / m_*^2 (1 + p_*^2 / m_*^2 + \Sigma_{5\alpha} \Sigma_5^\alpha / m_*^2)] / \mathcal{G} m_* f \\
p_\alpha^* f^\alpha &= [-2(p_\alpha^* \Sigma_5^\alpha / m_*^2)^2 + p_*^2 / m_*^2 (p_*^2 / m_*^2 - 1 + \Sigma_{5\alpha} \Sigma_5^\alpha / m_*^2)] / \mathcal{G} m_* f \\
\mathcal{G} &= p_*^2 / m_*^2 - 1 - \Sigma_{5\alpha} \Sigma_5^\alpha / m_*^2
\end{aligned}$$

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